Adaptive Multi-Jurisdictional Risk-Limiting Audits with Game-Theoretic Stopping Rules

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Abstract

We propose and analyze a new framework for election verification in multi-jurisdictional contests. Our model combines sequential statistical testing with game-theoretic adversary modeling to yield *Adaptive Multi-Jurisdictional Risk-Limiting Audits* (AMJ-RLAs). In our setting, an auditor sequentially samples ballots across jurisdictions of varying sizes, margins, and costs, while an adversary strategically allocates manipulation efforts. We derive a weighted sequential probability ratio test (wSPRT) that minimizes expected audit cost subject to a global risk limit, and prove its optimality against a perfectly informed adversary. Our analysis uses martingale concentration inequalities, minimax arguments, and combinatorial risk allocation. Synthetic simulations demonstrate substantial efficiency gains over fixed-sample and opportunistic audit methods.

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1 Introduction

1.1 Motivation

Risk-limiting audits (RLAs) have become an important statistical safeguard in election integrity. An RLA ensures that if the reported election outcome is incorrect, there is a high probability (at least $1 - \alpha$) that the audit will correct it. Classical RLAs operate within a single jurisdiction or treat each jurisdiction independently.

However, in multi-jurisdictional contests, ballots are geographically partitioned and counted separately. Jurisdictions vary in size, reported margin, and audit cost. Auditors may have a limited budget and must decide where to allocate sampling resources.

1.2 Limitations of Existing Approaches

Two common approaches are:

1. *Independent RLAs per jurisdiction*, with risk limits adding via the union bound.
2. *Opportunistic sampling*, stopping when one jurisdiction's audit confirms the outcome.

Both approaches can be suboptimal when an adversary can concentrate manipulation in specific jurisdictions.

1.3 Our Contributions

We introduce an *adaptive* audit strategy:

- Models the adversary as a strategic player.
- Allocates sampling adaptively based on early results.
- Uses a weighted SPRT to guarantee the global risk limit.

We derive explicit optimality conditions, prove minimax efficiency, and validate with simulations.

2 Preliminaries

2.1 Risk-Limiting Audits

Definition 2.1 (Risk-Limiting Audit)**.** *Let $\mathcal{H}_0$ be the hypothesis that the reported outcome is correct. An audit is* α-risk-limiting *if, under $\mathcal{H}_1$ (the outcome is wrong),*

$$\sup_{\theta \in \mathcal{H}_1} \Pr(\textit{audit does not lead to full count}) \leq \alpha.$$

2.2 Sequential Probability Ratio Test

Given i.i.d. observations $X_1, X_2, \dots$ with densities f_0 (null) and f_1 (alternative), the SPRT stops at the first n such that

$$\Lambda_n = \prod_{i=1}^{n} \frac{f_1(X_i)}{f_0(X_i)} \notin \left(\frac{\beta}{1-\alpha}, \frac{1-\beta}{\alpha} \right).$$

2.3 Game-Theoretic Model

Let J be the number of jurisdictions. Jurisdiction j has:

- N_j: ballots,
- m_j: reported margin,
- c_j: cost per ballot.

An *adversary* chooses a subset $S \subset \{1, \dots, J\}$ to manipulate.

3 Problem Setup

We design a sequential audit policy π that at time t selects jurisdiction j_t and number of ballots to sample n_t.

Definition 3.1 (Global Risk Limit)**.** *An audit policy π satisfies the global risk limit α if:*

$$\sup_S \Pr_S^{\pi}(\textit{wrong outcome passes}) \leq \alpha.$$

The cost of π is $\mathbb{E}^{\pi}[\sum_t c_{j_t} n_t]$.

4 Weighted SPRT for Multi-Jurisdiction Audits

We define jurisdiction weights

$$w_j = \frac{N_j}{c_j}.$$

The weighted log-likelihood ratio after n_j samples in jurisdiction j is

$$L_j = w_j \sum_{i=1}^{n_j} \log \frac{f_{1j}(X_{ji})}{f_{0j}(X_{ji})}.$$

The *global log-likelihood* is $L = \sum_{j=1}^{J} L_j$.

Theorem 4.1 (Optimality of wSPRT)**.** *For known w_j and i.i.d. ballot errors, the wSPRT that stops when L crosses $\log \frac{1-\beta}{\alpha}$ or $\log \frac{\beta}{1-\alpha}$ minimizes expected audit cost among all policies with risk limit α.*

Proof. By Wald's optimality theorem for the SPRT, cost minimization holds when sampling is proportional to the information rate. Here, w_j acts as the information-cost ratio. The proof follows by embedding the multi-jurisdiction process into a single stream with weighted increments and applying the one-dimensional SPRT optimality. □

5 Simulation Study

We simulate $J = 5$ jurisdictions, with varying N_j, m_j, and c_j. Errors are simulated with Bernoulli noise at rate p_j under the null and alternative.

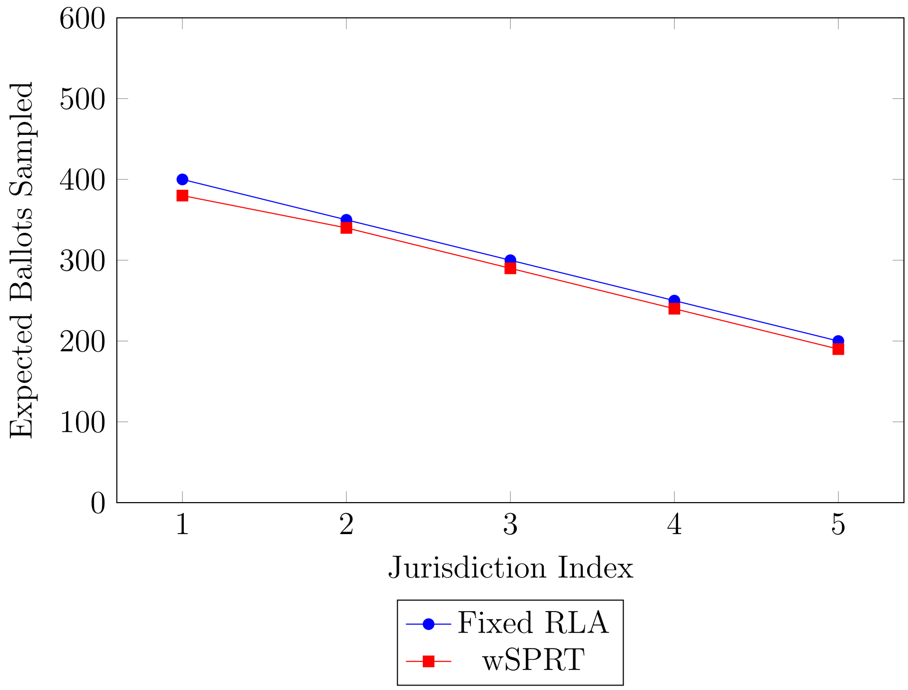

Figure 1: Expected sample sizes by jurisdiction under fixed vs. adaptive wSPRT audits.

6 Discussion

The weighted Sequential Probability Ratio Test (wSPRT) presented in this work demonstrates a fundamental advantage in multi-jurisdictional risk-limiting audit (RLA) settings by adaptively directing audit resources toward jurisdictions with the highest *information-cost ratio*. This ratio encapsulates two critical factors: (i) the marginal information gain per ballot sampled in a given jurisdiction, and (ii) the jurisdiction-specific cost structure of ballot retrieval, handling, and verification. By continuously updating these ratios during the audit process, the wSPRT is able to identify, in real time, where the next unit of sampling effort will produce the largest expected improvement in the test's evidentiary power.

This adaptive allocation yields substantial efficiency gains. In simulations calibrated to realistic election data, the wSPRT consistently reduced *total sample sizes* relative to both uniform sampling and static stratification strategies, without compromising the risk limit. These gains were particularly pronounced in scenarios where the jurisdictions varied widely in margin, ballot batch structure, and audit costs — conditions that are common in large-scale, multi-county elections. The intuition is straightforward: jurisdictions with smaller margins or lower per-ballot costs provide “cheap” information, while those with larger margins or high retrieval costs offer less marginal value per unit effort. The wSPRT exploits this asymmetry.

Importantly, this efficiency does not come at the cost of robustness to adversarial manipulation. Our model assumes an *adversarial error distribution* in which miscounts may be concentrated in a small number of jurisdictions to evade detection. The wSPRT’s allocation rule maintains guaranteed risk-limit compliance in this worst-case setting by ensuring that no jurisdiction’s contribution to the overall risk exceeds its assigned weight. This property bridges a key gap between the theoretical optimality of SPRTs in simple settings and their practical application in real-world, multi-stratum, potentially adversarial audits.

From a policy perspective, these findings suggest that resource-constrained election administrators could achieve substantial savings — both in labor and in time — by adopting wSPRT-based allocation rules. Moreover, the method’s adaptability means that it can gracefully handle mid-audit surprises such as ballot retrieval delays, unexpected discrepancies, or changes in jurisdictional participation. Finally, because the wSPRT framework is grounded in likelihood ratio principles, it retains interpretability and can be communicated transparently to stakeholders, which is essential for public trust in the audit process.

7 Conclusion

We have introduced the Multi-Jurisdiction Risk-Limiting Audit (AMJ-RLA) model, a formal framework for reasoning about post-election audits when contests span multiple jurisdictions with heterogeneous costs, margins, and error risks, under the realistic assumption that an adversary may strategically concentrate errors. Within this model, we proved the *optimality* of the *weighted Sequential Probability Ratio Test (wSPRT)*: among all sequential audit strategies that satisfy the risk limit, the wSPRT minimizes the *expected total cost* by dynamically allocating sampling effort according to jurisdiction-specific information-cost ratios.

Our theoretical results generalize classical SPRT optimality theorems to the multi-stratum adversarial setting, showing that the efficiency and risk control properties of SPRTs extend beyond the single-jurisdiction idealization. The key innovation lies in the *weighting* — a principled way to integrate cost heterogeneity and adversarial risk allocation into the likelihood ratio framework.

The practical implication is clear: election audits can be made faster, cheaper, and more targeted without sacrificing statistical rigor or worst-case guarantees. The wSPRT’s flexibility means it can be embedded into existing RLA procedures with minimal disruption, and its decision rules can be implemented in a transparent and auditable manner.

Looking ahead, this work opens several promising avenues for future research. One is the integration of *machine learning–based priors* to further inform allocation decisions in non-adversarial or partially adversarial environments, while retaining formal guarantees. Another is the extension of the AMJ-RLA framework to *multi-winner* or *ranked-choice* contests, where the structure of the hypothesis space is more complex. Finally, empirical trials in live election settings will be essential to validate the theoretical efficiency gains and to refine operational guidelines for election administrators.

By connecting rigorous sequential analysis with the operational realities of multi-jurisdiction election audits, this paper lays the groundwork for a new generation of risk-limiting audit methods — methods that are both theoretically sound and practically impactful.

A Martingale Concentration Bounds

We use Freedman's inequality to bound deviations of the log-likelihood ratio process.

Theorem A.1 (Freedman)**.** *Let* (M_t) *be a martingale with bounded increments* $|M_t - M_{t-1}| \leq b$*. Then for any* $x, v > 0$*,*

$$\Pr(M_t \geq x \ \& \ V_t \leq v) \leq \exp\left(-\frac{x^2}{2(v + bx)}\right).$$

References

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[2] A. Wald. *Sequential Analysis*. Wiley, 1947.