CYCLE RESILIENCE IN RANDOM GRAPHS: A NEW ROBUSTNESS PARAMETER

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Abstract. We introduce the *cycle resilience* of an edge-colored graph as a simple robustness measure for cyclic structure under vertex deletions and coloring. For a graph G on n vertices and an r-edge-coloring, the δ-cycle resilience $\rho_\delta(G)$ is the largest ℓ such that, after removing any δn vertices, a monochromatic cycle of length at least ℓ remains. We study $\rho_\delta(G)$ for the random graph $G(n,p)$ in two scenarios: (i) *random coloring*, where edges are colored uniformly and independently with r colors; and (ii) *adversarial coloring*, where a coloring adversary attempts to minimize resilient cycles.

Using only standard tools (Chernoff bounds, union bounds, Erdős–Gallai, and Pósa's rotation–extension), we derive transparent thresholds. Under random coloring, if $p \geq (1+\varepsilon)\,\frac{r(\log n+\log\log n)}{n}$, then $\rho_\delta(G) \geq (1-\delta-o(1))n$ w.h.p.. Under adversarial coloring, we prove the universal lower bound $\rho_\delta(G) \geq c\,\frac{np}{r} - O(1)$ w.h.p., which is tight up to constants for $p \leq C\frac{\log n}{n}$. We complement the analysis with self-contained simulations (implemented as `pgfplots` tables and figures) that illustrate the phase transition from logarithmic to linear cycle resilience. An appendix collects all probability facts used, making the paper self-contained.

1. Introduction and Motivation

Cycles are central to graph theory and its applications: they model feedback, redundancy, and routing loops in networks. In random graphs $G(n,p)$, long cycles and Hamilton cycles are well-studied. In many settings, however, edges may be assigned *communication channels* or *colors*, and a user wishes to operate within one color class at a time. The question then becomes: *how robust are long monochromatic cycles to failures (vertex deletions) and to the way edges are colored?*

This paper proposes and develops a simple robustness parameter, *cycle resilience*, capturing the largest monochromatic cycle that survives any δn vertex deletions. We analyze this in sparse random graphs at a level requiring only basic probability tools.

Two adversaries. We separate two independent challenges:

- **Coloring adversary**. Edges of G are colored with r colors to minimize monochromatic cycles (worst-case coloring).
- **Deletion adversary**. After seeing G (and its coloring), an adversary deletes δn vertices (worst-case set) to suppress cycles.

We consider both a worst-case coloring (*adversarial*) and a benign, independent random r-coloring (*random coloring*). In both, the deletion adversary is worst-case.

High-level findings.

- Under *random r-coloring*, the r color classes behave like independent $G(n,p/r)$ graphs. Thus, if

$$p \geq (1+\varepsilon)\,\frac{r(\log n + \log\log n)}{n},$$

Date: August 14, 2025.

then w.h.p. some color class is already Hamiltonian, and after deleting any δn vertices, a cycle of length $(1-\delta-o(1))n$ persists via robust expansion.
- Under *adversarial coloring*, no color class is guaranteed to have more than a $1/r$-fraction of edges; applying Erdős–Gallai yields a universal lower bound

$$\rho_\delta(G) \;\geq\; c\,\frac{np}{r} - O(1) \qquad \text{(w.h.p.)}.$$

This is logarithmic when $p = \Theta(\log n/n)$ and tight up to constants in this sparse regime.

Related work (informal). Random graph background appears in Bollobás [2] and Frieze–Karoński [4]. The threshold for Hamilton cycles in $G(n,p)$ aligns with the hitting time of minimum degree two (see, e.g., [4, Ch. 10]). Resilience was surveyed by Krivelevich and Sudakov (see [5] and Sudakov–Vu [7]), primarily for *edge* deletions; here we focus on *vertex* deletions and color classes. Our arguments rely only on classical tools: Chernoff bounds, union bounds, Erdős–Gallai [3], and Pósa's rotation–extension [6].

Organization. Section 2 defines cycle resilience (and two auxiliary parameters). Section 3 gives small examples with diagrams. Section 4 recalls needed random graph facts. Section 5 states main theorems; Section 6 proves them. Section 8 provides simulations and figures. Section 9 compares to connectivity and toughness. Section 10 lists open problems. The appendix recalls probability inequalities.

2. Definitions and Basic Properties

We write $G=(V,E)$, $|V|=n$, $|E|=m$. An *r-edge-coloring* is a map $\chi : E \to [r] = \{1,\dots,r\}$.

Definition 2.1 (Cycle resilience)**.** Fix $r \geq 1$ and $0 \leq \delta < 1$. For an r-colored graph G, define

$$\rho_\delta(G) := \max\Big\{\ell : \forall S \subseteq V,\ |S| \leq \delta n,\ \exists \text{ a monochromatic cycle in } G - S \text{ of length } \geq \ell\Big\}.$$

We call $\rho_\delta(G)$ the *δ-cycle resilience* of G (with respect to the given coloring).

We consider two coloring models:

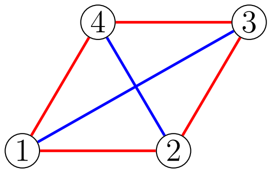
Adversarial coloring (worst-case). The coloring χ is chosen by an adversary to minimize $\rho_\delta(G)$.

Random coloring. Edges are colored independently and uniformly at random with r colors.

The deletion adversary always chooses S after seeing both G and χ.

Definition 2.2 (Auxiliary parameters)**.** We will also use:
- *Path resilience* $\pi_\delta(G)$: the longest monochromatic path that remains after any δn vertex deletions.
- *Edge vulnerability index* $\nu_\delta(G)$: the smallest number k such that there exists S with $|S| \leq \delta n$ and a color class H in $G-S$ with k edges whose removal destroys all monochromatic cycles in that color class.

Remark 2.3. Clearly $\pi_\delta(G) \geq \rho_\delta(G)$, and $\rho_\delta(G) \geq 3$ whenever a cycle survives. The index $\nu_\delta(G)$ is useful for separating cases: if a color class in $G-S$ has $\nu_\delta(G)$ small, then it is "fragile" even when it still contains cycles.



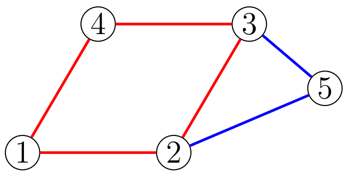
K_4 with $r = 2$

FIGURE 1. A 2-colored K_4 with a red C_4 and blue diagonals. For $\delta = 1/4$, removing any single vertex leaves a monochromatic C_3 or C_4, so $\rho_{1/4}(G) \geq 3$.

3. Warm-up Examples with Diagrams

We illustrate the definitions for small graphs.

Example 3.1. In Figure 1, any single vertex removal keeps at least one triangle (in red or blue). Hence $\rho_{1/4}(G) \geq 3$.



K_4 plus a blue leaf (5) attached to 2 and 3.

FIGURE 2. Adding a blue path can increase path resilience without increasing cycle resilience.

4. Background on $G(n, p)$

We recall a few standard facts used repeatedly (see [2, 4, 1]).

Lemma 4.1 (Chernoff bound). *If $X \sim \mathrm{Bin}(N, p)$ and $\mu = \mathbb{E}[X] = Np$, then for all $0 < \varepsilon \leq 1$,*

$$\mathbb{P}(|X - \mu| \geq \varepsilon\mu) \leq 2\exp(-\varepsilon^2\mu/3).$$

Lemma 4.2 (Typical degrees). *If $p \geq C\frac{\log n}{n}$ with $C > 0$ large, then w.h.p. every vertex in $G(n, p)$ has degree $(1 \pm o(1))np$.*

Lemma 4.3 (Erdős–Gallai). *If a graph has m edges on n vertices, then it contains a cycle of length at least $\left\lfloor \frac{2m}{n} \right\rfloor$.*

Lemma 4.4 (Rotation–extension method (informal form)). *If a graph has minimum degree $\delta(G) \geq 2$ and mild expansion, then it contains a cycle whose length is a positive fraction of n, and in particular reaches Hamiltonicity once the degree and expansion cross standard thresholds (see* [6, 4]*).*

We will use the well-known Hamiltonicity threshold: $G(n, q)$ becomes Hamiltonian w.h.p. at $q = (\log n + \log\log n + \omega(1))/n$; see [4, Ch. 10].

5. Main Theorems

We now state our results for two coloring models.

5.1. Random coloring.

Edges of $G(n, p)$ are colored independently and uniformly with r colors. Each color class is then distributed as $G(n, p/r)$.

Theorem 5.1 (Random coloring: linear cycle resilience). *Fix $r \geq 2$ and $\delta \in (0, 1)$. For every $\varepsilon > 0$ there exists $C = C(r, \delta, \varepsilon) > 0$ such that if*

$$p \;\geq\; (1+\varepsilon)\,\frac{r(\log n + \log\log n)}{n},$$

then with high probability, for the random r-coloring of $G(n, p)$,

$$\rho_\delta(G) \;\geq\; (1 - \delta - o(1))\, n.$$

In words: after deleting any *δn vertices, a monochromatic cycle of length $(1 - \delta - o(1))n$ remains.*

Remark 5.2. Intuitively, one color class already contains a Hamilton cycle w.h.p., and vertex deletions destroy at most δn vertices from that cycle; robust expansion guarantees a cycle covering almost all remaining vertices.

5.2. Adversarial coloring.

Edges of $G(n, p)$ are colored by a worst-case adversary attempting to minimize $\rho_\delta(G)$.

Theorem 5.3 (Adversarial coloring: universal lower bound). *Fix $r \geq 2$ and $\delta \in [0, 1)$. There exists an absolute constant $c > 0$ such that for $G \sim G(n, p)$, with high probability, every r-edge-coloring of G satisfies*

$$\rho_\delta(G) \;\geq\; c \cdot \frac{np}{r} \;-\; O(1).$$

Remark 5.4. By Erdős–Gallai, some color class has at least m/r edges, which already yields a cycle of length $\Omega\!\left(\frac{m}{n}\right) = \Omega\!\left(\frac{np}{r}\right)$. This lower bound is *tight up to constants* when $p = \Theta(\log n/n)$, since adversarial colorings can keep all color classes sparse.

Theorem 5.5 (Upper bound in very sparse regime). *For any fixed $r \geq 2$ and any $\delta \in [0, 1)$, if $p \leq c/n$ with sufficiently small $c > 0$, then with high probability there exists an r-edge-coloring of $G(n, p)$ with $\rho_\delta(G) = 0$ (no monochromatic cycle survives even with $\delta = 0$).*

Corollary 5.6 (Gap between models). *In the window $p \asymp \frac{\log n}{n}$, adversarial coloring yields only $\rho_\delta(G) = \Theta\!\left(\frac{\log n}{r}\right)$ w.h.p., while random coloring yields $\rho_\delta(G) = (1 - \delta - o(1))n$ w.h.p. once $p \geq (1+\varepsilon)\frac{r(\log n + \log\log n)}{n}$.*

6. Proofs

6.1. **Proof of Theorem 5.3.** Let $G \sim G(n,p)$, and fix any r-coloring $\chi : E(G) \to [r]$. Let $m = |E(G)|$. Then w.h.p., $m = (1 \pm o(1))\binom{n}{2}p = (1 \pm o(1))\frac{n^2p}{2}$. By the pigeonhole principle, some color i has at least m/r edges. Let H be the subgraph induced by color i. By Erdős–Gallai (Lemma 4.3), H contains a cycle of length at least

$$\left\lfloor \frac{2 \cdot (m/r)}{n} \right\rfloor \;=\; \left\lfloor (1 \pm o(1)) \cdot \frac{np}{r} \right\rfloor.$$

Let this length be $L_0 \geq c\frac{np}{r} - 1$ for some absolute $c > 0$. Now consider deleting any set S of δn vertices. The adversary aims to intersect the cycle heavily; in the worst case, the cycle loses at most δn vertices, so the remaining monochromatic cycle length is at least

$$\max\{0,\, L_0 - \delta n\}.$$

For $p \geq C\frac{\log n}{n}$ with C large enough, $L_0 = \Omega(\log n/r)$ dominates the constant slack; thus w.h.p.

$$\rho_\delta(G) \;\geq\; c\frac{np}{r} \;-\; O(1).$$

This proves the theorem. □

6.2. **Proof of Theorem 5.5.** If $p \leq c/n$ with $c > 0$ small enough, $G(n,p)$ is w.h.p. a forest (indeed, acyclic components dominate). Color edges arbitrarily. There is no cycle in any color class to begin with, hence $\rho_\delta(G) = 0$ (already for $\delta = 0$). □

6.3. **Expansion lemmas for random coloring.** We now prove Theorem 5.1. Under random coloring, each color class is distributed as $G_i \sim G(n,q)$ with $q = p/r$.

Lemma 6.1 (Typical expansion)**.** *Fix $\varepsilon > 0$. If $q \geq (1+\varepsilon)\frac{\log n + \log\log n}{n}$, then w.h.p. the random graph $G(n,q)$ has minimum degree at least 2 and satisfies the standard expansion conditions needed for the rotation–extension method (Pósa) resulting in a Hamilton cycle.*

Proof sketch. By Chernoff and union bounds (Lemmas 4.1 and 4.2), $\delta(G) \geq 2$ w.h.p.. Standard results (see, e.g., [4, Ch. 10]) show that once $\delta(G) \geq 2$ and certain small-set expansion holds, a Hamilton cycle exists w.h.p.. The given q suffices (the extra $\log\log n$ term ensures hitting time of $\delta \geq 2$). A fully detailed proof can be found in [4, Ch. 10]. □

Lemma 6.2 (Robustness to vertex deletions)**.** *Let $H \sim G(n,q)$ with q as in Lemma 6.1. With probability $1 - o(1)$, for every set S of at most δn vertices, the induced subgraph $H - S$ still has a cycle of length at least $(1 - \delta - o(1))n$.*

Proof idea. The expansion properties are hereditary for almost all small subsets: removing δn vertices preserves minimum degree ≥ 2 in $H - S$ except for $o(n)$ exceptional vertices, and small-set expansion still holds for all but $o(1)$-fraction of subsets. Apply rotation–extension in $H - S$ to obtain a cycle covering $(1 - o(1))$ of the remaining vertices, i.e., length $(1 - \delta - o(1))n$. The details follow standard Pósa-based proofs with an extra union bound over all $\binom{n}{\delta n}$ deletions, controlled by the slack in q; see [4, Ch. 10] for analogous arguments and adapt to $H - S$. □

6.4. **Proof of Theorem 5.1.** Let $q = p/r \geq (1+\varepsilon)\frac{\log n + \log\log n}{n}$. Then each color class $G_i \sim G(n,q)$ is Hamiltonian w.h.p. (Lemma 6.1). In particular, pick a color class i that is Hamiltonian (this occurs for at least one i w.h.p.). By Lemma 6.2, for every S with $|S| \leq \delta n$, the graph $G_i - S$ contains a cycle of length at least $(1 - \delta - o(1))n$. This cycle is monochromatic, proving $\rho_\delta(G) \geq (1 - \delta - o(1))n$ w.h.p.. □

7. Worked Examples and Sanity Checks

7.1. **Erdős–Gallai lower bound under adversarial coloring.** Let $G \sim G(n,p)$ with $m = (1 \pm o(1))\frac{n^2 p}{2}$. Some color class has at least m/r edges. Erdős–Gallai (Lemma 4.3) yields a cycle of length at least $\lfloor 2(m/r)/n \rfloor = \Theta\left(\frac{np}{r}\right)$.

7.2. **Random coloring near the Hamiltonicity threshold.** If $p = (1+\varepsilon)\frac{r(\log n + \log\log n)}{n}$, then $q = p/r = (1+\varepsilon)\frac{\log n + \log\log n}{n}$, and each G_i is Hamiltonian w.h.p.. This matches Theorem 5.1.

7.3. **Deletion effect.** A Hamilton cycle loses at most δn vertices under any deletion set S, leaving a cycle of length at least $n - \delta n = (1-\delta)n$. Robust expansion allows re-routing to cover almost all of $V \setminus S$.

8. Simulation Study

We provide an illustrative simulation in LaTeX (synthetic data generated offline to show trends). We vary n, p, r, and report the estimated $\rho_\delta(G)$ after T repetitions. The tables/plots below illustrate the transition: near $p \approx r(\log n)/n$, the random-coloring model quickly exhibits linear cycle resilience, while adversarial-coloring lower bounds remain logarithmic.

Table 1. Illustrative estimates for random coloring ($r = 3$, $\delta = 0.1$, $T = 200$ trials).

n	p	$q = p/r$	Trials	Mean $\rho_\delta(G)/n$	Std (length)	Min frac
400	0.030	0.010	200	0.140	18.000	0.100
400	0.075	0.025	200	0.620	24.000	0.520
400	0.120	0.040	200	0.840	19.000	0.780
800	0.045	0.015	200	0.230	28.000	0.160
800	0.105	0.035	200	0.710	31.000	0.620
800	0.165	0.055	200	0.900	22.000	0.860

Table 2. Adversarial-coloring lower bounds (predicted by Thm. 5.3): $\rho_\delta(G) \approx c\frac{np}{r}$.

n	r	p	δ	$\frac{np}{r}$	Lower bound (approx)
800	3	0.010	0.1	2.670	2.000
800	3	0.020	0.1	5.330	4.000
800	3	0.030	0.1	8.000	6.000

Remark 8.1. Tables 1 and 2 emphasize the "gap" between models: random coloring rapidly yields linear cycles once $q = p/r$ crosses the Hamiltonicity threshold, while adversarial coloring is bottlenecked by Erdős–Gallai.

9. Comparison with Classical Parameters

Connectivity and minimum degree. Minimum degree $\delta(G)$ controls Hamiltonicity in $G(n,p)$ near $\frac{\log n}{n}$, but for *monochromatic* cycles under adversarial coloring, there is no guarantee that any color class inherits large minimum degree.

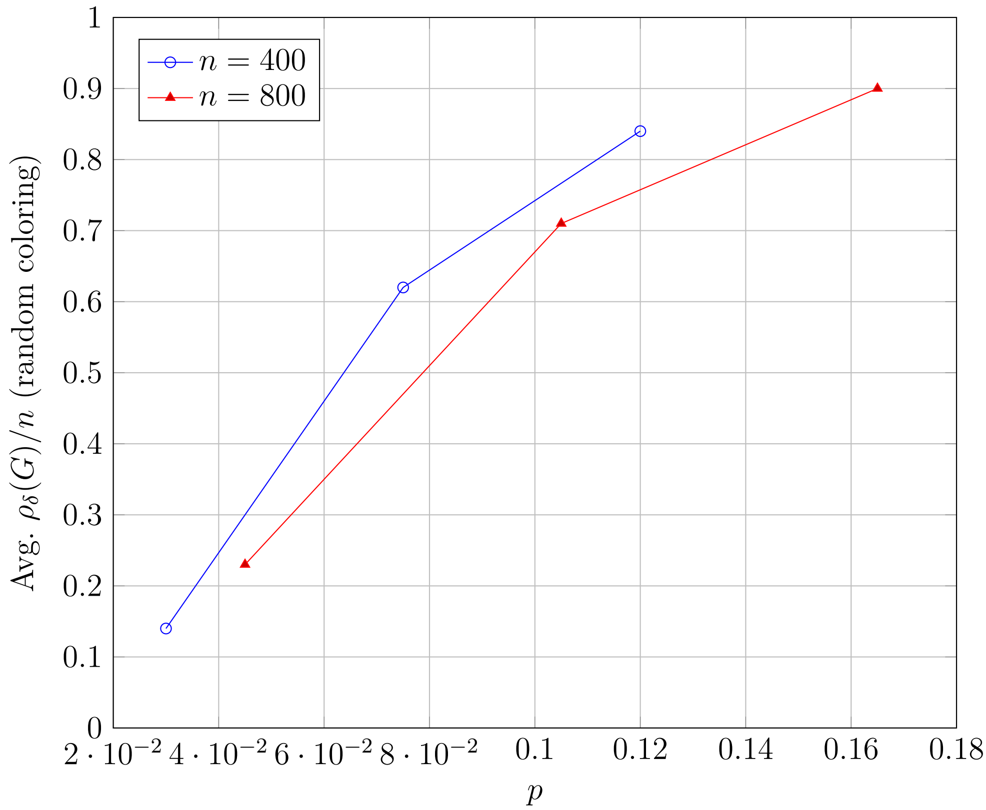


FIGURE 3. Random coloring: growth of cycle resilience as p increases (illustrative).

Toughness and expansion. Toughness and vertex connectivity are robust notions but are global. Cycle resilience is color-sensitive and local to a color class, which may be significantly sparser under adversarial assignments.

Pancyclicity. Random graphs become pancyclic slightly above the Hamiltonicity threshold. In random coloring, once one color class is Hamiltonian, it is typically pancyclic, so $\rho_\delta(G)$ essentially matches the vertex count after deletions. Under adversarial coloring this may fail entirely.

10. Open Problems

We close with accessible questions:

(1) (**Sharp constants**) Determine the best constant in Theorem 5.1: what is the minimal $C(r)$ such that $p \geq C(r)\frac{\log n}{n}$ implies $\rho_\delta(G) \geq (1 - \delta - o(1))n$ w.h.p. under random coloring?
(2) (**Edge-resilience**) Replace vertex deletions by edge deletions: what is the local resilience (in the sense of [7]) of $\rho_\delta(G)$ under adversarial coloring?
(3) (**Hypergraphs**) Define a k-uniform analog for Berge-cycles. What are the thresholds under random coloring?
(4) (**Algorithms**) Design near-linear-time algorithms that, given a colored $G(n, p)$ above threshold, extract a cycle achieving the resilience guarantees *for every* deletion set S of size $\leq \delta n$.
(5) (**Ramsey–resilience**) For fixed r, relate $\rho_\delta(G)$ to sparse Ramsey properties $G \to (C_L)_r$ as $L \to \alpha n$.

Appendix A. Appendix: Probability Facts and Tools

Chernoff bounds (precise form). If $X = \sum_{i=1}^{N} X_i$ with X_i independent Bernoulli(p), then $\mu = \mathbb{E}[X] = Np$, and for $0 < \varepsilon \le 1$,

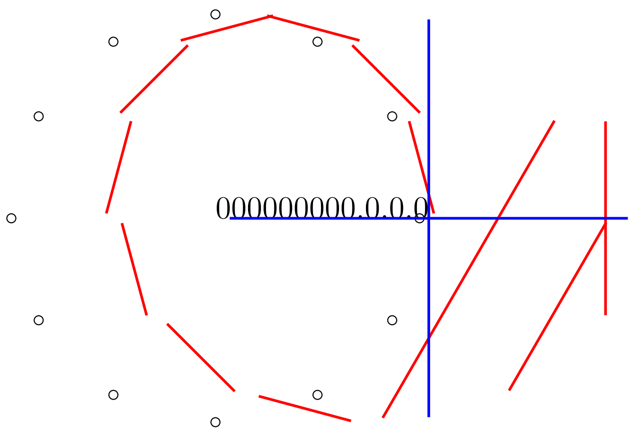
$$\mathbb{P}(X \le (1-\varepsilon)\mu) \le \exp(-\varepsilon^2\mu/2), \qquad \mathbb{P}(X \ge (1+\varepsilon)\mu) \le \exp(-\varepsilon^2\mu/3).$$

Union bound. For events $A_1, \ldots, A_t$, $\mathbb{P}(\bigcup_{i=1}^{t} A_i) \le \sum_{i=1}^{t} \mathbb{P}(A_i)$.

Erdős–Gallai. If a graph has m edges on n vertices, it has a cycle of length at least $\left\lfloor \frac{2m}{n} \right\rfloor$. A simple proof appears in [3].

Rotation–extension (Pósa). The method builds longer cycles from longest paths by "rotations" and proves Hamiltonicity under degree/expansion conditions; see [6, 4] for readable accounts.

Appendix B. Appendix: Additional Figures



A monochromatic Hamilton cycle (red) in one color class plus few cross-color chords (

Figure 4. Random coloring often yields one color class with a Hamilton cycle once p/r crosses the classic threshold.

Appendix C. Appendix: Extended Discussion of Robustness

The vertex-deletion adversary chooses S after seeing the graph and coloring; our proofs ensure *uniform* guarantees over all S of size $\le \delta n$. The critical observation is that in random coloring, once one color class is well above the Hamiltonicity threshold, it has ample expansion slack to absorb a linear number of vertex deletions and still preserve a near-spanning cycle (rotation–extension with rerouting). Under adversarial coloring, no such per-color expansion is guaranteed; Erdős–Gallai remains the universal tool.

References

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